Practice Midterm Exam

This exam is closed-book and closed-computer. You may have a double-sided, 8.5” × 11” sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices (laptops, cell phones, etc.) during the course of this exam. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 48 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to check your understanding of the topics we’ve covered so far. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

You have three hours to complete this exam. There are 48 total points.

Question	**Points**	**Graders**
(1) Mathematical Logic	/ 12	
(2) Set Theory	/ 12	
(3) Equivalence Relations	/ 12	
(4) Graphs and Induction	/ 12	
	/ 48	

<u>You can do this.</u> Best of luck on the exam!

Problem One: Mathematical Logic (12 Points)

(Midterm Exam, Fall 2018)

An ***autocontrapositive*** is an implication that is its own contrapositive (after a little bit of simplification). For example, if you start with the implication

$$P(x) \to \neg P(x)$$

and take its contrapositive, you get

$$\neg\neg P(x) \to \neg P(x),$$

which simplifies down to

$$P(x) \to \neg P(x),$$

which is the same implication we started with!

i. **(1 Point)** In the space below, we've written the first half of an autocontrapositive. Fill in the blank to complete the autocontrapositive, and do so in a way that ***does not use*** the ¬ connective. No justification is necessary.

$$(\forall x.\ \neg P(x)) \quad \to \quad \underline{\hspace{8cm}}$$

ii. **(2 Points)** In the space below, we've written the first half of an autocontrapositive. Fill in the blank to complete the autocontrapositive, and do so in a way that ***does not use*** the ¬ connective. No justification is necessary.

$$(\exists x.\ (P(x) \land \forall y.\ (Q(x) \to \neg R(y)))) \quad \to \quad \underline{\hspace{8cm}}$$

iii. **(3 Points)** In the space below, we've written the first half of an autocontrapositive. Fill in the blank to complete the autocontrapositive, and do so in a way that ***does not use*** the ¬ connective. No justification is necessary.

$$((\exists x.\ (\neg P(x) \leftrightarrow \neg Q(x))) \leftrightarrow \exists y.\ \neg S(y)) \quad \to \quad \underline{\hspace{8cm}}$$

(This question is independent of the earlier parts of this problem.)

If you're looking for a next book to read, may I suggest checking out the works of Joan Didion? She's written some real masterpieces in *Slouching Toward Bethlehem* and *The Year of Magical Thinking*. Although Joan Didion is already fairly well-known, I still think she deserves wider recognition.

iv. **(6 Points)** Using the predicates

- $Person(p)$, which states that p is a person;
- $BookByDidion(b)$, which states that b is a book by Joan Didion;
- $Read(x, y)$, which states that x has read y; and
- $Knows(x, y)$, which states that x knows y,

write a statement in first-order logic that says "there is a person who has read every book by Joan Didion, but doesn't know anyone else who's read even a single one of Joan Didion's books."

Problem Three: Proofwriting, Set Theory (8 Points)

(Midterm Exam, Spring 2017)

On Problem Set One and Problem Set Two, you gained experience writing rigorous mathematical proofs about sets and set theory. In this problem, we're going to ask you to prove two results about sets. The results here are actually quite useful and show up in later CS courses, like CS243 (where they're used in the design of optimizing compilers) and CS258 (where they're used to mathematically model computer programs).

i. **(4 Points)** Prove that for any set S, if $A \in \wp(S)$ and $B \in \wp(S)$, then $A \cup B \in \wp(S)$.

   In the course of writing up your proof, please call back to the formal definitions of the relevant set relations and operations.

The following interaction is often used in conjunction with the previous result.

ii. **(4 Points)** Prove that for any sets A and B, if $A \cup B \subseteq B$, then $A \subseteq B$.

In the course of writing up your proof, please call back to the formal definitions of the relevant set relations and operations.

Problem Three: Equivalence Relations (12 Points)

(Midterm Exam, Winter 2018)

On Problem Set Three, you explored equivalence relations and a number of their properties. This problem explores the interaction of equivalence relations with some terminology that, up to this point, we've more typically associated with functions.

Let's begin with a new definition. A binary relation R over a set A is called ***surjective*** if the following statement is true about R:

$$\forall b \in A.\ \exists a \in A.\ aRb.$$

This definition is closely related to the definition of surjectivity for functions, hence the name.

Prove that a binary relation R over a set A is an equivalence relation if and only if R is surjective, symmetric, and transitive.

Problem Four: Graphs and Induction (12 Points)

(Midterm Exam, Winter 2018)

On Problem Set Four, you explored different properties of graphs and how to write proofs about them. On Problem Set Five, you practiced writing proofs by induction. This problem is designed to let you demonstrate what you've learned in the course of doing so.

Let's begin with a new definition. A ***triangle-free graph*** is one that contains no triangles (as a refresher, a ***triangle*** is a collection of three mutually adjacent nodes). That is, if you pick three distinct nodes in the graph, some two of them will not be adjacent.

Prove by induction that if G is a triangle-free graph with $2n$ nodes (for some natural number n), then G has at most n^2 edges. Some hints:

- Remember that $(n + 1)^2 = n^2 + 2n + 1$.
- What happens if you pick a pair of *adjacent* nodes in a triangle-free graph and delete them? Think about trying to count up how many edges there are in the graph after you do this.